The Range of Realization

Melvin Fitting
City University of New York

Vienna — July 2014

Justification Logics (Background)

Justification Logics are like modal logics, except they involve *explicit reasoning* within the language itself.

You will see examples shortly.

Let's start with some history.

Intuitionistic logic
was intended to be constructive.

The well-known
Brouwer, Heyting, Kolmogorov
(BHK)
semantics has a constructive flavor.

It is based on
an abstract notion of *proof*.

$\perp$ has no proof.

A proof of $X \wedge Y$ consists of
a proof of X and a proof of Y.

A proof of $X \vee Y$ consists of
a proof of X or a proof of Y.

A proof of $X \supset Y$ consists of
an algorithm converting any proof of X
into a proof of Y.

But, what is a proof?

Can this be given an
arithmetic interpretation?

In 1933 Gödel made a first step.

One can characterize
intuitionistic “truth”
using classical validity plus
informal provability.

Gödel proposed that
informal provability should
meet the following conditions
(writing $\Box X$ for X is "provable").

classical tautologies

$$\Box(A \supset B) \supset (\Box A \supset \Box B)$$

$$\Box A \supset A$$

$$\Box A \supset \Box\Box A$$

$\vdash A$ and $\vdash A \supset B$ implies $\vdash B$

$\vdash A$ implies $\vdash \Box A$

This is the well-known modal logic S4.

Translate intuitionistic formulas
by putting $\Box$ before every subformula.

For example,
$$(A \wedge B) \supset A$$
becomes $\Box((\Box(\Box A \wedge \Box B) \supset \Box A)$

Then, X is an intuitionistic theorem
if and only if the translate of X
is a theorem of S4.

Gödel was an expert on
embedding logic into arithmetic.

He noted that S4
does *not* embed into arithmetic.

At least not by using
his provability predicate
$(\exists y)(y$ is the Gödel number of a proof of $x)$
to interpret $\Box$.

In 1938 Gödel had another proposal,
interpret $\Box$ as *explicit* provability.

(This moves the existential quantifier to the metalevel.)

This was not published during Gödel's lifetime.

The idea was independently rediscovered
by Sergei Artemov in the 1990's.

Artemov introduced a logic *LP*
(logic of proofs)

We will see the details shortly.

It is a kind of *explicit* modal logic.

This will mean something shortly.

The Basic Picture

$$\mathsf{Int} \hookrightarrow \mathsf{S4} \hookrightarrow \mathsf{LP} \hookrightarrow \mathsf{Arith}$$

Gödel's translation

Realization Theorem (Artemov)

Arithmetic Embedding Theorem (Artemov)

Intuitionistic logic has an arithmetic interpretation.

The Basic Picture

$$\mathsf{Int} \hookrightarrow \mathsf{S4} \hookrightarrow \mathsf{LP} \hookrightarrow \mathsf{Arith}$$

Realization
Theorem
(Artemov)

What we concentrate on.

And what is LP?

The really new things are
proof terms

(now usually called
justification terms)

Variables, v_1, v_2, $\ldots$ are proof terms.

Constant symbols, c_1, c_2, $\ldots$ are proof terms.

If t and u are proof terms, so are $t + u$ and $t \cdot u$.

If t is a proof term, so is $!t$.

Formulas are built up from
propositional letters, P, Q, $\ldots$, and $\bot$.

Using $\supset$
and maybe other connectives.

And, if t is a proof term,
and X is a formula, $t{:}X$
is a formula.

Think of $t{:}X$ as asserting:
X is so, with t as a proof,
or t is a justification for X.

The informal ideas:

$t \cdot u$ justifies X whenever
u justifies some formula Y,
and t justifies $Y \supset X$.

$t + u$ justifies X whenever
t justifies X,
or u justifies X.

If t justifies X,
$!t$ justifies that fact.

Constants justify formulas that
we do not further analyze;
that is, axioms.

Variables stand for
arbitrary justifications.

LP Axioms

A0.	**Classical**	Tautologies
A1.	**Application**	$t{:}(X \supset Y) \supset (s{:}X \supset (t \cdot s){:}Y)$
A2.	**Factivity**	$t{:}X \supset X$
A3.	**Justification Checker**	$t{:}X \supset\, !t{:}(t{:}X)$
A4.	**Weakening**	$s{:}X \supset (s+t){:}X$
		$t{:}X \supset (s+t){:}X$

LP Rules

R1.	**Modus Ponens**	$\vdash Y$ provided $\vdash X$ and $\vdash X \supset Y$
R2.	**Axiom Necessitation**	$\vdash c{:}X$ where X is an axiom *A0* – *A4* and c is a justification constant.

Note to self: say something about Constant Specifications.

What follows is an abbreviated example of a proof in LP.

1. $x{:}P \supset (x{:}P \vee y{:}Q)$

2. $a{:}(x{:}P \supset (x{:}P \vee y{:}Q))$ using axiom nec.

3. $a{:}(x{:}P \supset (x{:}P \vee y{:}Q)) \supset (!x{:}x{:}P \supset [a\cdot!x]{:}(x{:}P \vee y{:}Q))$

4. $!x{:}x{:}P \supset [a\cdot!x]{:}(x{:}P \vee y{:}Q)$

5. $x{:}P \supset !x{:}x{:}P$

6. $x{:}P \supset [a\cdot!x]{:}(x{:}P \vee y{:}Q)$

7. $y{:}Q \supset [b\cdot!y]{:}(x{:}P \vee y{:}Q)$ similarly

8. $x{:}P \supset [a\cdot!x + b\cdot!y]{:}(x{:}P \vee y{:}Q)$ weakening

9. $y{:}Q \supset [a\cdot!x + b\cdot!y]{:}(x{:}P \vee y{:}Q)$ similarly

10. $(x{:}P \vee y{:}Q) \supset [a\cdot!x + b\cdot!y]{:}(x{:}P \vee y{:}Q)$

So we have

$$(x{:}P \vee y{:}Q) \supset [a{\cdot}!x + b{\cdot}!y]{:}(x{:}P \vee y{:}Q)$$

where a justifies the tautology

$$x{:}P \supset (x{:}P \vee y{:}Q)$$

and b justifies the tautology

$$y{:}Q \supset (x{:}P \vee y{:}Q)$$

Internalization

We have that
if X is an axiom,
$a{:}X$ for some constant a.

In fact, if X is a theorem,
$t{:}X$ for some justification term t.

The structure of t
internalizes the proof of X.

What is Realization?

For any LP formula X
let X° be the result of
replacing every justification term
with $\Box$.

This is the
forgetful functor.

If X is an LP theorem,
X° is an S4 theorem.

True for axioms.

For example,

$$s{:}(X \supset Y) \supset (t{:}X \supset [s \cdot t]{:}Y)$$

becomes

$$\Box(X \supset Y) \supset (\Box X \supset \Box Y).$$

The rules preserve
this property.

That's all there is
to this.

But a converse also holds!

If X is a theorem of S4,
there is a theorem Y of LP
so that $Y^\circ = X$.

Better yet, Y can have
distinct justification variables
where X has negative $\Box$.

Positive $\Box$ occurrences become
terms computed from these variables.

There is a kind of
input/output structure.

This is called a
normal realization.

For example, the S4 theorem

$$(\Box P \vee \Box Q) \supset \Box(\Box P \vee \Box Q)$$

has the normal realization

$$(x{:}P \vee y{:}Q) \supset [a\cdot !x + b\cdot !y]{:}(x{:}P \vee y{:}Q).$$

S4 can be thought of as
a logic of knowledge
(with positive introspection).

Then

$$(KP \vee KQ) \supset K(KP \vee KQ)$$

says something about our
implicit knowledge.

$$(x{:}P \vee y{:}Q) \supset [a{\cdot}!x + b{\cdot}!y]{:}(x{:}P \vee y{:}Q)$$

makes reasoning about our knowledge *explicit*.

If we have a reason for one of P or Q here is how to go about verifying that fact.

It's a Family!

I've talked about
modal S4 and justification LP.

But it turns out that
a large number of modal logics
have justification counterparts.

Examples are K, T, D, K4, D4.

Since these are sublogics of S4,
one just omits parts of the
S4 and LP machinery.

Then counterparts are connected
via realization theorems.

S5 was an early example
that needed new justification machinery
(an additional function symbol).

But recently I've realized,
the family of modal logics
with justification counterparts
is very big.

Let's look at S4.2 as an example.

Axiomatically, add to S4,

$$\Diamond\Box X \supset \Box\Diamond X$$

or equivalently,

$$\Box\neg\Box X \vee \Box\neg\Box\neg X$$

Semantically, use S4 frames
that are *convergent*.

$u\mathcal{R}v_1$ and $u\mathcal{R}v_2$ implies
there is some w with
$v_1\mathcal{R}w$ and $v_2\mathcal{R}w$.

For a justification counterpart,
add to LP two new function symbols,
and the axiom

$$f(t,u){:}\neg t{:}X \vee g(t,u){:}\neg u{:}\neg X$$

Let's call this logic J4.2

Here's an informal
plausibility argument.

$$\neg t{:}X \vee \neg u{:}\neg X$$

is provable in LP.

In any context one of the disjuncts must hold.

$$f(t,u){:}\neg t{:}X \vee g(t,u){:}\neg u{:}\neg X$$

says we can compute a justification for whichever does hold.

J4.2 realizes S4.2

For example, here is an S4.2 theorem:

$$[\Diamond\Box A \wedge \Diamond\Box B] \supset \Diamond\Box(A \wedge B)$$

Or equivalently,

$$[\neg\Box\neg\Box A \wedge \neg\Box\neg\Box B] \supset \neg\Box\neg\Box(A \wedge B)$$

It is realized by

$$\{\neg[j_4 \cdot j_3 \cdot !v_5 \cdot g(!v_3, j_2 \cdot v_5 \cdot !v_9)]{:}\neg v_9{:}A \wedge$$
$$\neg[j_5 \cdot f(!v_3, j_2 \cdot v_5 \cdot !v_9)]{:}\neg v_3{:}B\} \supset \neg v_5{:}\neg[j_1 \cdot v_9 \cdot v_3]{:}(A \wedge B)$$

where:

$$\begin{aligned}
&j_1{:}\{A \supset (B \supset (A \wedge B))\}\\
&j_2{:}\{\neg[j_1 \cdot v_1 \cdot v_3]{:}(A \wedge B) \supset (v_1{:}A \supset \neg v_3{:}B)\}\\
&j_3{:}\{v_5{:}\neg[j_1 \cdot v_1 \cdot v_3](A \wedge B) \supset \{\neg[j_2 \cdot v_5 \cdot v_2]{:}\neg v_3{:}B \supset \neg[v_2 \cdot v_1]{:}A\}\}\\
&j_4{:}\{\neg !v_9{:}v_9{:}A \supset \neg v_9{:}A\}\\
&j_5{:}\{\neg !v_3{:}v_3{:}B \supset \neg v_3{:}B\}
\end{aligned}$$

Realization
How Proved?

There are many different proofs of Realization.

I prefer a two-stage version.

Sometimes constructive.
Needs a cut-free modal
proof procedure.

Stage 1: produce a *quasi-realization*
from a modal validity.
Stage 2: convert the quasi-realization
to a realization.

Stage 1: produce a *quasi-realization* from a modal validity.
Stage 2: convert the quasi-realization to a realization.

Always constructive.
Algorithm is independent of the particular logic.

Algorithms for quasi-realizations need:

A formal proof in a proof system that

1. has the subformula property
2. preserves subformula polarity

What has worked so far:

1. Sequent calculus
2. Tableaus
3. Nested Sequents
4. Prefixed tableaus
5. Hypersequents

Many modal logics don't have such proof systems.
(Or so it seems.)

There is also a semantic proof of quasi-realization.
This is not constructive.
But it is more general.
I'll sketch the idea.

Modal Models
(very familiar)

$$\mathcal{M} = \langle \underbrace{\mathcal{G}, \mathcal{R}}_{\text{frame}}, \mathcal{V} \rangle$$

atomic truth assignment

$\mathcal{M}, \Gamma \Vdash \Box X$ if
$\mathcal{M}, \Delta \Vdash X$ whenever
$\Gamma \mathcal{R} \Delta$

Justification Models
(Fitting Models)

$$\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$$

Evidence function

Note to self: please explain!

$$\Gamma \in \mathcal{E}(t, X)$$

Γ is a world at which
t is relevant evidence
for X.

$$\mathcal{E}(t, X) \cap \mathcal{E}(s, X \supset Y) \subseteq \mathcal{E}(s \cdot t, Y)$$

$$\mathcal{E}(s, X) \cup \mathcal{E}(t, X) \subseteq \mathcal{E}(s + t, X)$$

$$\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$$

$\mathcal{M}, \Gamma \Vdash t{:}X$ if

$\mathcal{M}, \Delta \Vdash X$ whenever $\Gamma \mathcal{R} \Delta$

and

$\Gamma \in \mathcal{E}(t, X)$

For Example

For LP, corresponding to S4:

Frame is transitive and reflexive.

$\mathcal{E}$ is monotonic:

$$\frac{\Gamma \mathcal{R} \Delta}{\Gamma \in \mathcal{E}(t, X) \Longrightarrow \Delta \in \mathcal{E}(t, X)}$$

$$\mathcal{E}(t, X) \cap \mathcal{E}(s, X \supset Y) \subseteq \mathcal{E}(s \cdot t, Y)$$

$$\mathcal{E}(s, X) \cup \mathcal{E}(t, X) \subseteq \mathcal{E}(s + t, X)$$

$$\mathcal{E}(t, X) \subseteq \mathcal{E}(!t, t{:}X)$$

Completeness, how proved

Axiomatic completeness for justification logics is, so far, by a canonical model construction.

Let's use LP as an example.

$\mathcal{G}$ is all maximally LP consistent sets.

For $\Gamma \in \mathcal{G}$, $\Gamma^\sharp = \{X \mid t{:}X \in \Gamma\}$.

$\Gamma \mathcal{R} \Delta$ if $\Gamma^\sharp \subseteq \Delta$.

$\Gamma \in \mathcal{E}(t, X)$ if $t{:}X \in \Gamma$.

$\Gamma \in \mathcal{V}(X)$ if $X \in \Gamma$.

Now, prove the usual
Truth Lemma.

And show $\langle \mathcal{G}, \mathcal{R} \rangle$
is an S4 frame:
reflexive and transitive.

Key step.
Justification terms must
"fit together" correctly
for this to happen.

The Current State of Things

I'll describe where
we are so far.

What works for LP and S4
can be generalized.

Suppose ML is a canonical *modal* logic.

Suppose JL is a candidate for a *justification* counterpart.

Suppose the canonical justification model for JL is built on a frame for ML.

Then a Realization Theorem connects ML and JL.

For Example

Geach formulas,

$\Diamond^k \Box^l A \supset \Box^m \Diamond^n A$,

where $k, l, m, n \geq 0$.

Equivalently,

$\Box^k \neg \Box^l A \vee \Box^m \neg \Box^n \neg A$

Logics axiomatized using Geach formulas:

D,

T,

B,

K4,

S4,

S4.2,

S5

Theorem: Any modal logic axiomatized by Geach formulas has a justification logic counterpart, with a connecting Realization theorem.

My guess is that
this extends to Sahlquist formulas,
but I don't know yet.

It is clear that
justification logics are a general phenomenon.

How general is only beginning to be clear.

And that's where things are now.

A Few References

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Thank You